

Simultaneous Straight-line Drawing of a Planar Graph and Its Rectangular Dual

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Abstract. A natural way to represent on the plane both a planar graph and its dual is to follow the definition of the dual, thus, to place vertices inside their corresponding primal faces, and to draw the dual edges so that they only cross their corresponding primal edges. The problem of constructing such drawings has a long tradition when the drawings of both primal and dual are required to be straight-line. We consider the same problem for a planar graph and its rectangular dual. We show that the rectangular dual can be resized to host a planar straight-line drawing of its primal.

1 Introduction

A *planar drawing* of a planar graph is its representation on the plane such that its vertices are mapped to distinct points and its edges to non-intersecting simple Jordan curves. A drawing is called *straight-line* if each edge is represented by a line segment. It is well-known that each planar graph admits a planar straight line drawing [14], even with a quadratic area [10,23]. A planar drawing Γ partitions the plane into topologically connected regions called *faces*, the unbounded face is called *external* and the remaining are called *internal* faces. The edges that bound the external face are also called *external*, the remaining edges are *internal*. A *planar embedding* of a planar digraph G is an equivalence class of planar drawings that induce the same clockwise cyclic ordering of edges around each vertex and that have the same external face.

An alternative way to represent a planar graph G is to draw the vertices as geometric shapes so that, two shapes touch¹ if and only if the corresponding vertices of G are adjacent. Such type of representation is called *contact representation*. Different kinds of shapes for contact representations of planar graph have been considered (eg., [1,11,12,17]). One of the most simple for the visual perception is a contact representation with rectangles. A contact representation with rectangles is called *rectangular subdivision*, if it forms a partition of a rectangle into a set of smaller non-intersecting rectangles such that no four of them meet at the same point. Such contact representation of a planar graph G is known as a *rectangular dual* of G , and is denoted by D . Figure 1 shows a planar graph and its rectangular dual. Graph G is referred to as *primal* of D . Unfortunately, not

¹ We say that two shapes touch if they have a common interval of a positive length.

all planar graphs admit a rectangular dual. In particular, a planar graph G has a rectangular dual D with four rectangles on the boundary if and only if every internal face of G is a triangle, the external face is a quadrangle and there is no separating triangles in G (see eg., [19, Theorem 2.1]). The condition that D is bounded by four rectangles can be relaxed [18].

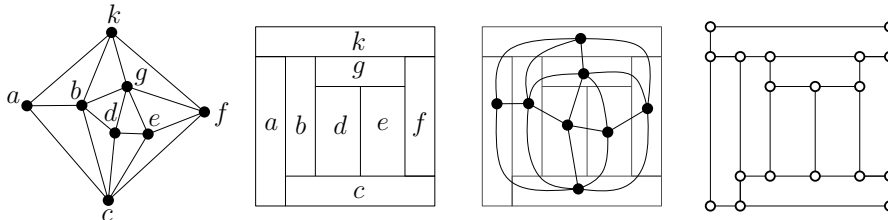


Fig. 1. From left to right: A graph G , its rectangular dual D , a simultaneous planar-rectangular dual drawing of G and D , graph G' constructed from D .

A natural way to simultaneously represent both a planar graph G and its rectangular dual D is to draw a vertex of G inside its corresponding rectangle and to draw each edge (u, v) as a curve crossing only the common segment of the rectangles representing u and v (see Figure 1, second from the right). To simplify the visual complexity of such a representation one may ask for a straight-line drawing of G . Such a drawing is called *straight-line simultaneous drawing* of a graph G and its rectangular dual D . It is not surprising that if the drawing of a rectangular dual D is fixed, then it might not be possible to position the vertices of G in order to obtain a straight-line simultaneous drawing of G and D [2, Lemma 1]. It is also known that the corresponding decision problem is \mathcal{NP} -hard [2, Theorem 1]. In this paper, we show that if the rectangular dual D is allowed to be resized, i.e. we are allowed to change the sizes of the rectangles (without further changing the structure) we can achieve a straight-line simultaneous drawing of the primal graph G and of this resized rectangular dual.

The problem of straight-line simultaneous drawing of a graph and its rectangular dual finds application in visualization of clustered graphs [2,20]. A rectangular subdivision can be seen as a simplification of a map. Assume that each region of this map contains some elements related to each other (a cluster, and intra-cluster edges) and to the elements of adjacent regions (inter-cluster edges). Some possible readability requirements for a visualization of this network together with the map are that the entire network is drawn in a planar fashion, the intra-cluster edges must lie completely inside the corresponding region and the inter-cluster edges must cross only the common segment of the regions where their end-points lie. A simple approach to construct such a visualization is to contract each cluster to a vertex, which results in the graph primal to the given rectangular subdivision. Then, construct a straight-line simultaneous drawing of

the primal and the rectangular dual and, finally, uncontract the clusters. This approach is described in detail in [2, Theorem 2].

Allow us, until the end of this section, to reverse the roles of G and D . In particular, consider the graph G' , the vertices of which are the corners of the rectangles of D and the edges of which are the parts of the sides of the rectangles connecting the vertices (see Figure 1, right). The edges of this graph are represented either by horizontal or by vertical segments, and each face, including the external, forms a rectangle, i.e. D implies a so-called *rectangular drawing* of G' [22]. Then, graph G becomes a *week dual*² of this new graph G' . When asking for a straight-line simultaneous drawing of G and D we are actually asking for the straight-line simultaneous drawing of the primal G' and its dual G . This point of view helps us to summarize the related work in the next paragraphs.

Drawings of the primal and dual graphs so that each vertex of the dual is placed inside the corresponding face of the primal and each dual edge crosses only the corresponding primal edge will be referred to as *simultaneous planar-dual drawing*. Such a drawing is called *straight-line* if both graphs are drawn straight-line. It is immediately clear that in case of a non-week dual a straight-line simultaneous planar-dual drawing does not exist and at least one edge (dual to an external edge) need to have a bend. To avoid this special case from now on we only consider the week dual graph, without further mentioning it.

Already back in 1963 Tutte [24] considered the problem of constructing a straight-line simultaneous planar-dual drawing and showed that it exists when the primal graph is triconnected. The drawing constructed by Tutte's algorithm may have exponentially large area. Only four decades later, Erten and Kobourov [13] provided a linear-time algorithm to construct a straight-line simultaneous planar-dual drawing on a grid of size $(2n-2) \times (2n-2)$ for the same family of graphs. Later, Zhang and He [25] improved this result to a grid of size $(n-1) \times n$.

Observe that talking about straight-line simultaneous planar-dual drawing, we ask for the construction of the drawings of both the primal and the dual. A stricter variation of the straight-line simultaneous planar-dual drawings was considered by Bern and Gilbert [6], where the drawing of the primal graph is fixed and one only need to determine the positions of dual vertices. Notice that here it is also required that each dual edge crosses only the corresponding primal. Bern and Gilbert observed that the problem is easy if all faces of the primal are triangles, thus the dual vertices can be placed at the meeting points of angle bisectors. They presented a linear time algorithm to construct the drawing of the dual in case where convex quadrilateral faces are also present. They showed that a straight-line drawing of the dual does not always exist if non-convex

² The *dual graph* G^* of a planar graph G with a fixed planar drawing is formed by placing a vertex inside each face of G , and connecting vertices of G^* whose corresponding faces in G are adjacent. A *week dual* of the graph G results by removing from G^* the vertex representing the external face of G . Graph G is called the *primal* of G^* .

quadrilaterals are present. Finally, they proved that the decision problem is \mathcal{NP} -hard for five-sided convex faces.

Specially convenient for a discretization method [8] are straight-line simultaneous planar-dual drawings with the additional requirement that the primal and dual edges cross at right-angle. Such drawing always exists if each internal face of the primal graph is a non-obtuse triangle. In particular, the dual graph can be drawn by joining perpendicular bisectors of the edges [5]. The requirement of right-angle crossing in straight-line simultaneous planar-dual drawings have also been studied in [3,9,21], see [7] for an overview.

In this paper we show that a given rectangular dual D of a planar graph G can be resized so that G and this resized rectangular dual obtain a simultaneous straight-line drawing. Proof of this statement is the subject of Section 4. Before proving this result we introduce the necessary definitions in Section 2 and give preliminary observations in Section 3.

2 Definitions and useful facts

st-digraph and its parts Consider now a directed graph G , *digraph*, for short. A *source* (resp. a *sink*) of G is a vertex with only outgoing (resp. incoming) edges. An *st-digraph* is an acyclic digraph with exactly one source s and exactly one sink t . A *planar st-digraph* is an *st-digraph* that is planar and provided with a planar embedding such that vertices s and t lie on the boundary of the external face (Figure 2, left). It is common to visualize planar *st-digraphs* in an *upward* fashion, i.e. with edges represented by curves monotonically increasing in upward direction.

It is not hard to see (refer also to [4, Lemma 4.1]) that a face f of a planar *st-digraph* G is bounded by two directed paths meeting only at the source and at the sink of f (see Figure 2, left). If we imagine G being embedded upward we can characterize these paths as the *left* and the *right boundaries* of f . Let e be an edge of G , the face of G lying to the left (resp. right) of e is called *left* (resp. *right*) face of e (see again Figure 2, left).

Topological ordering and dual digraph A *topological ordering* of a digraph $G = (V, E)$ is a 1-1 function $\rho : V \rightarrow \{1 \dots |V|\}$ such that for every edge (u, v) we have $\rho(u) < \rho(v)$ (Figure 2, left).

We define the *dual digraph* G^* of a planar *st-digraph* G as follows. The vertex set of G^* is the set of internal faces of G plus the two vertices, s^* and t^* , for the external face of G , where s^* is for its left and t^* for its right boundary (Figure 2, right). For every edge $e \neq (s, t)$ of G , G^* has an edge $e^* = (f, g)$ where f and g are the left and the right faces of e , respectively. Digraph G^* is generally a multigraph, but in this work we merge the multiple edges to one. Digraph G^* is an *st-digraph* with the source s^* the sink t^* [4].

Rectangular dual Let G be a graph. A *rectangular dual* D of G is a rectangular subdivision \mathcal{R} and a one-to-one correspondence between the vertices of G and

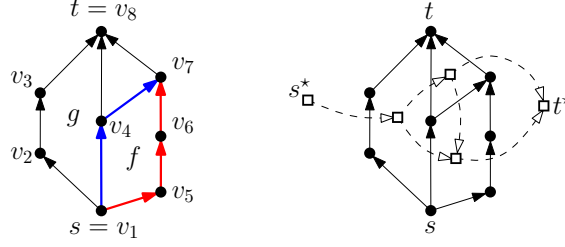


Fig. 2. Left: A planar st -digraph G . The blue (resp. red) path comprises the left (resp. right) boundary of face f . Face g (resp. f) is the left (resp. right) face of the edge (s, v_4) . Indices of the vertices are given accordingly to a topological ordering of the vertices. Right: The dual G^* of the st-digraph G is depicted by dashed edges, the multiple edges are merged.

the rectangles of \mathcal{R} such that two vertices are adjacent in G if and only if their corresponding rectangles share a common boundary. If two vertices are adjacent we say that the corresponding rectangles are also *adjacent*. For the sake of simplicity we use the same notation for the vertices of G and for the rectangles of D .

Simultaneous drawing of a planar graph and its rectangular dual Let G be a graph admitting a rectangular dual D . We say that G and D have a *straight-line simultaneous drawing*, if we can place each vertex of G inside its corresponding rectangle of D such that if the edges of G are drawn straight-line, the resulting drawing of G is planar and each edge (u, v) crosses D only at a single point contained in the common boundary of the rectangles representing u and v .

Notation and operations for rectangles and rectangular dual Let u be a rectangle on the plane with edges parallel to coordinate axes. We denote by $x_1(u)$, $x_2(u)$, $y_1(u)$, $y_2(u)$ the x- and y-coordinates of the corners of u , where $x_1(u) < x_2(u)$ and $y_1(u) < y_2(u)$. We denote by $R(u)$ the rightmost segment of u . Let v be a different rectangle adjacent to u . We denote by $[u, v]$ the maximal common segment of u and v . If segment $[u, v]$ is vertical, we denote by $y_1[u, v]$, $y_2[u, v]$ the y-coordinates of its end-points. We say that u and v have *vertical* (resp. *horizontal*) adjacency if the segment $[u, v]$ is vertical (resp. horizontal).

Given a rectangle u on the plane, we define as *stretch* of u as the increase of $x_2(u)$. Let D be a rectangular dual of a planar graph G . We define a *scaling* of D to be a rectangular dual of G that results from resizing of some of the rectangles of D . Observe that scaling does not change the type of the adjacency (vertical or horizontal) of two adjacent rectangles of D .

Visibility Our target is to construct a straight-line simultaneous drawing of a planar graph G and its rectangular dual D . When we place vertex u inside its corresponding rectangle in D we denote by $x(u)$, $y(u)$ the coordinates of

u . Our main requirement is that an edge (u, v) of G crosses the boundaries of the rectangles of D only at a single point and particularly it only crosses the segment $[u, v]$. To work with this requirement, we need several additional definitions which reflect the notion of visibility of a vertex inside a rectangle to an adjacent rectangle. Let u and v be two adjacent vertices of G , and assume that the position of vertex u in its corresponding rectangle of D is fixed. The *visibility region* of u inside v , denoted by $\text{vis}(u, v)$, is the region delimited by the boundary of the rectangle v and the lines through u and the two end-points of the segment $[u, v]$, see for example Figure 3 and Figure 4. If u and v have horizontal adjacency such that u is above v and $x_1(u) < x_1(v)$, $x_2(u) = x_2(v)$ then $\text{vis}(u, v)$ contains the topmost segment of $R(v)$ (Figure 3). In case u and v have vertical adjacency we distinguish two types of visibility region as follows. We say that the visibility region $\text{vis}(u, v)$ is *diverging* and that u is a *diverging neighbor* of v if the two lines through u delimiting $\text{vis}(u, v)$ have slopes of a different sign. Figure 4, depicts several cases of diverging visibility regions. Figure 5, top left, depicts non-diverging visibility regions.

Assume again that u and v have a vertical adjacency and u is to the left of v . In case u is a non-diverging neighbor of v we have that either (1) $y_1(v) < y_1(u) < y_2(v) < y(u) < y_2(u)$ or (2) $y_1(u) < y(u) < y_1(v) < y_2(u) < y_2(v)$. See Figure 5, top left for the illustration of the case (1). Consider the set $R(v) \setminus \text{vis}(u, v)$, it is non-empty and generally contains two segments; one containing the topmost point of $R(v)$ and one containing the bottommost point of $R(v)$. We denote by $\text{blind}(u, v)$ the segment of $R(v) \setminus \text{vis}(u, v)$ which contains the topmost (resp. bottommost) point of $R(v)$ for the case (1) (resp. (2)) (Figure 5, top left).

During the construction of the straight-line simultaneous drawing of G and its rectangular dual D we mostly place vertices close to the right boundary of the rectangles. To formalize this we use the following notation. Let u be a rectangle, we denote by $\text{gate}(u)$ a proper sub-interval of $R(u)$ not containing both end-points of $R(u)$.

Regular edge labeling Let G be a planar embedded planar graph with no separating triangle, with exactly four vertices on the external face and each internal face being a triangle. It is known that such a graph has a rectangular dual D (see e.g. [18]). Two adjacent rectangles of D have either vertical or horizontal adjacency. This fact is mirrored by so-called *regular edge labeling* (REL, for short) [18], defined for graph G , which is also known as *transversal structure* [15]. It is formally defined as follows. A REL of G is a partition and orientation of its interior edges resulting in two disjoint sets of arcs E^R and E^B , so that:

- For each internal vertex u the edges incident to u appear in the counterclockwise order as follows: edges in E^R outgoing from u , edges in E^B incoming to u , edges in E^R incoming to u , and edges in E^B outgoing from u ; moreover, none of these four sets of edges is empty;
- Four outer vertices of G are named v_N , v_S , v_W , and v_E . Moreover, the internal edges incident to v_S (resp. v_N) are all in E^R and are outgoing from v_S (resp. incoming to v_N). Also, the internal edges incident to v_W (resp. v_E) are all in E^B and are outgoing from v_W (resp. incoming to v_E).

It is known that every planar graph without separating triangles, with exactly four vertices on the outer face and each internal face being a triangle has a REL [18, Theorem 2.2]. Such a REL is used as a tool for constructing a rectangular dual of G . Let G^R (resp. G^B) be the directed subgraph of G induced by the edges in E^R (resp. E^B) and the four exterior edges directed such that v_S (resp. v_W) is a source of G^R (reps. G^B) and v_N (resp. v_E) is a sink of G^R (reps. G^B). We will heavily rely on the fact that G^R is a planar st -digraph with source v_S and sink v_N [18, Lemma 2.3]. We use *red* and *blue* colors to distinguish edges in E^R and E^B .

Observe that given a rectangular dual D of G , one can construct a REL by: (a) placing internal edges of G depicting horizontal adjacency to E^R and orienting them from bottom to top, and (b) placing internal edges depicting vertical adjacency to E^B and orienting them from left to right. This REL is said to be *defined by* the rectangular dual D . The reverse is also true, given a REL of G one can construct a rectangular dual D such that the blue edges specify vertical and the red edges horizontal adjacency [18, Theorem 4.3]. This D is said to be *consistent* with the given REL.

3 On visibility between two adjacent rectangles

The following statement is illustrated in Figure 3.

Statement 1 *Let u and v be two horizontally adjacent rectangles, such that u is above v , $x_1(u) < x_1(v)$ and $x_2(u) = x_2(v)$. There exists $X \geq x_2(v)$ such that, $\forall x \geq X$, if we set $x_2(v) = x_2(u) = x$ then $\text{gate}(v) \subset \text{vis}(u, v)$.*

Proof. The statement follows from the facts that (1) the region $\text{vis}(u, v)$ contains the upper part of $R(v)$ and (2) the lower half-line delimiting the region $\text{vis}(u, v)$ has a negative slope.

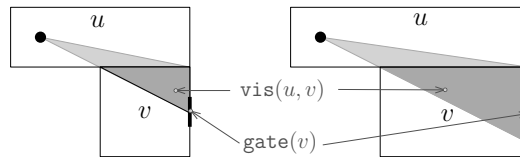


Fig. 3. Two horizontally adjacent rectangles such that $x_1(u) < x_1(v)$ and $x_2(u) = x_2(v)$, the case considered in Statement 1. The fact $\text{gate}(v) \subset \text{vis}(u, v)$ does not hold before, but holds after the stretch of u and v .

Statement 2 *Let u and v be two vertically adjacent rectangles, such that u is to the left of v and u is a diverging neighbor of v . There exists $X \geq x_2(v)$ such that, $\forall x \geq X$, if we set $x_2(v) = x$ then the $\text{gate}(v) \subset \text{vis}(u, v)$.*

Proof. The four cases determined by possible relations among the coordinates $y_1(u)$, $y_2(u)$, $y_1(v)$ and $y_2(v)$ are shown on Figure 4. Since the region $\text{vis}(u, v)$ is diverging, i.e. the half-lines through u delimiting $\text{vis}(u, v)$ have positive and negative slope, there exists $X \geq x_2(v)$, such that, for any $x \geq X$, if we set $x_2(v) = x$, then $R(v) \subset \text{vis}(u, v)$ and therefore $\text{gate}(v) \subset \text{vis}(u, v)$.

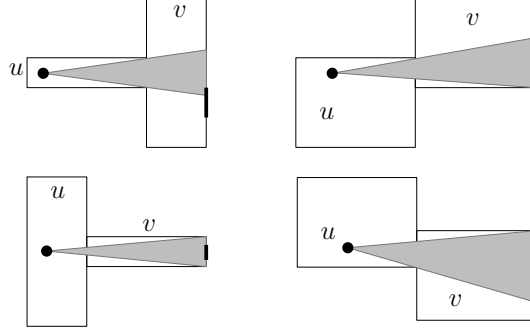


Fig. 4. The illustration of Statement 2. Four cases of vertical adjacency where u is a diverging neighbor of v .

Statement 3 *Let u and v be two vertically adjacent rectangles, such that u is to the left of v and $y(u) > y_2[u, v]$ (u is a non-diverging neighbor of v). If $\text{gate}(v) \cap \text{blind}(u, v) = \emptyset$ then there exists $X \geq x_2(v)$ such that, $\forall x \geq X$, if we set $x_2(v) = x$ then $\text{gate}(v) \cap \text{blind}(u, v) \neq \emptyset$. Otherwise, there exists $y < y(u)$ such that if we set $y_2(v) = y$ then $\text{gate}(v) \subset \text{vis}(u, v)$.*

Proof. Assume first that $\text{gate}(v) \cap \text{blind}(u, v) = \emptyset$ (see Figure 5, top left) for the illustration. Since both lines delimiting $\text{vis}(u, v)$ have negative slope, as $x_2(v)$ grows $\text{gate}(v)$ “slides” over $\text{vis}(u, v)$ from bottom to top. Thus, there exists $X \geq x_2(v)$ such that, $\forall x \geq X$, if we set $x_2(v) = x$ then $\text{vis}(u, v) \cap \text{blind}(u, v) \neq \emptyset$ (Figure 5, top right).

Assume now that $\text{gate}(v) \cap \text{blind}(u, v) \neq \emptyset$ (Figure 5, bottom left). Observe that as $y_2(v)$ increases and remains less than $y(u)$, the slope of the topmost half-line delimiting $\text{vis}(u, v)$ increases and remains negative. Thus, if $y_2(v) = y(u)$, then the mentioned half-line has slope zero and $\text{blind}(u, v) = \emptyset$. So, when $y_2(v)$ tends to $y(u)$, $\text{blind}(u, v)$ tends to \emptyset . Recall that $\text{blind}(u, v)$ is the topmost segment of $R(v)$, and that $\text{gate}(v)$ does not contain the topmost point of $R(v)$. Hence, there exists $y_2(v) < y < y(u)$ when $\text{blind}(u, v)$ is small enough and does not intersect with $\text{gate}(v)$ (Figure 5, bottom right).

The following statement is symmetric to Statement 3 and can be proven identically.

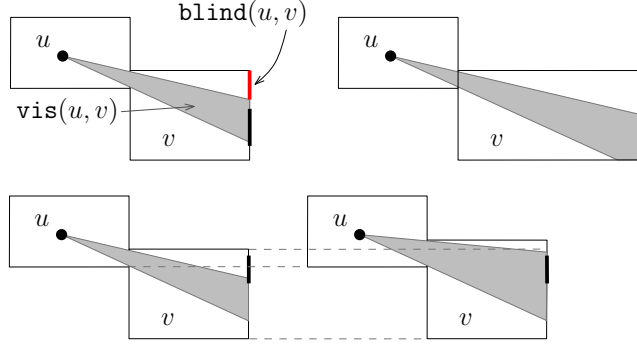


Fig. 5. The top left figure illustrates the case when $\text{gate}(v) \cap \text{blind}(u, v) = \emptyset$. The top right figure shows that after stretching v we get that $\text{gate}(v) \cap \text{blind}(u, v) \neq \emptyset$. The second line illustrates what happens when $y_2(v)$ is increased. In the left figure we have that $\text{gate}(v) \cap \text{blind}(u, v) \neq \emptyset$. The right figures shows the situation after the increase of $y_2(v)$, we get that $\text{gate}(v) \subset \text{vis}(u, v)$.

Statement 4 *Let u and v be two vertically adjacent rectangles, u is to the left of v and $y(u) < y_1[u, v]$ (u is a non-diverging neighbor of v). If $\text{gate}(v) \cap \text{blind}(u, v) = \emptyset$ then there exists $X \geq x_2(v)$ such that, $\forall x \geq X$, if we set $x_2(v) = x$ then the $\text{gate}(v) \cap \text{blind}(u, v) \neq \emptyset$. Otherwise, there exists $y > y(u)$ such that if we set $y_2(v) = y$ then $\text{gate}(v) \subset \text{vis}(u, v)$.*

4 Main result

Theorem 1. *Let G be a planar graph admitting a rectangular dual D . There exists a scaling D' of D such that G and D' admit a straight-line simultaneous drawing.*

Proof. We assume that D is bounded by four rectangles (see Figure 6) that have horizontal adjacency between each other, if not so we add them to D , as well as the corresponding vertices to G . For the simplicity of notation we denote the new graph by G and its rectangular dual by D . After the scaling D' of D is created and the straight-line simultaneous drawing of G and D' is constructed we simply remove the added vertices and rectangles. We denote the bottommost rectangle of D by v_S , the topmost by v_N , the leftmost by v_W and the rightmost by v_E .

Let E be the edge set of G and let E^R, E^B be the REL of G that is defined by rectangular dual D (refer to Figure 6). Recall that G^R is the subgraph of G containing only the edges of E^R plus the four external edges that are oriented so that v_S is a source and v_N is a sink. Recall also that G^R is an st -digraph.

Let $(G^R)^*$ be the dual of G^R , it is an st -digraph by itself. Let f_1, \dots, f_k be a topological ordering of the vertices of $(G^R)^*$ (see Figure 7). We denote by G_i^R , $1 \leq i \leq k$ the subgraph of G^R constituted by the vertices and edges of the faces

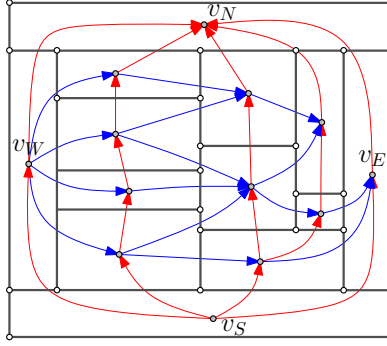


Fig. 6. Rectangular drawing Γ of the primal graph H , which we treat as a contact of rectangles. The dual G of H is colored according to its REL. This graph is used as an example throughout the paper.

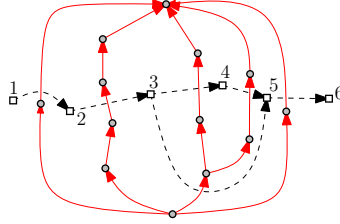


Fig. 7. The graph G^R , its dual $(G^R)^*$ and a topological ordering of $(G^R)^*$, which in this case is unique.

f_1, \dots, f_i (Figure 8, left). As a special case, graph G_1^R is the subgraph containing only vertices v_S, v_W, v_N and the edges $(v_S, v_W), (v_W, v_N)$. While $G_k^R = G^R$. It is not difficult to see that G_i^R is an st -digraph and that G_{i+1}^R can be constructed from G_i^R by adding the right boundary of f_{i+1} to the external face of G_i^R (see Figure 8). This fact is proven formally in [16, Lemma 4] for maximal planar st -digraphs, the proof for non-maximal planar st -digraph is along the same lines. Let G_i be the subgraph of G that is induced by the vertices of G_i^R (see Figure 9). Observe that the edges of G_i that do not belong to G_i^R are blue and lie in the internal faces of G_i^R .

In the following, by induction on i , $1 \leq i \leq k$, we construct a rectangular dual D_i of G_i consistent with the REL E^R, E^B (restricted to G_i), and show how to construct a straight-line simultaneous drawing of G_i and D_i . For $i = k$ we will obtain a rectangular dual D' of $G_k = G$ and the straight-line simultaneous drawing of G and D' . Since D' is consistent with the REL E^R, E^B it represents a scaling of D . So the theorem will follow.

Base case: $i = 1$. G_1 consists of the vertices v_W, v_S, v_N . The graph G_1 contains only the red edges (v_S, v_W) and (v_W, v_N) . We represent the three vertices v_S, v_W, v_N of R_1 as three rectangles such that $x_1(v_S) = x_1(v_W) = x_1(v_N)$,

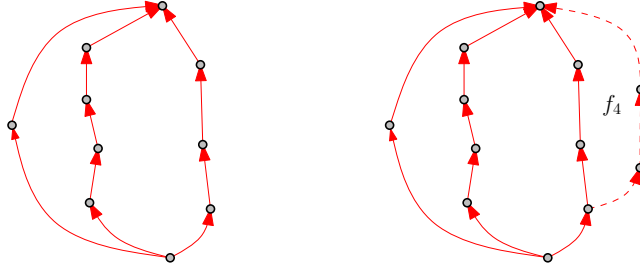


Fig. 8. Left: The graph G_3^R . Right: The graph G_4^R , which is produced from G_3^R by adding the vertices and the edges of the right boundary of f_4 to the external face of G_3^R .

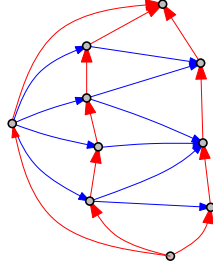


Fig. 9. The graph G_3 .

$x_2(v_S) = x_2(v_W) = x_2(v_N)$ and $y_2(v_S) = y_1(v_W) < y_2(v_W) = y_1(v_N)$ (see Figure 10). Graph G_1 contains exactly the same edges as G_1^R and a straight-line simultaneous drawing of G_1 and D_1 can be constructed trivially.

Induction hypothesis. For each $j \leq i < k$, there exists a rectangular dual D_j of G_j consistent with (G^R, G^B) such that G_j and D_j have a straight-line simultaneous drawing.

Induction step. As mentioned above, G_{i+1}^R can be constructed from G_i^R by adding the right boundary of f_{i+1} to the external face of G_i^R . Let directed path u_1, \dots, u_a be the left boundary of f_{i+1} . Since vertices u_1, \dots, u_a represent a directed sub-path of the right boundary of the external face of G_i^R , the rectangles u_1, \dots, u_a appear on the right boundary of D_i and lie on top of each other. See Figure 10 for a specific example. Let $u_1, v_1, \dots, v_b, u_a$ be the right boundary of f_{i+1} . We stretch all the rectangles of the right boundary of D_i except for u_2, \dots, u_{a-1} by the same value (Figure 10, right) and place the new rectangles v_1, \dots, v_b vertically between u_1 and u_a , according to the adjacency between the vertices u_2, \dots, u_{a-1} and v_1, \dots, v_b . We set $x_2(v_1) = \dots = x_2(v_b) = x_2(u_a)$, thus the current D_{i+1} is bounded by a rectangle and obviously comprises a rectangular dual of G_i , consistent with $\text{REL } E^R, E^B$.

Consider those of rectangles (resp. vertices) v_1, \dots, v_b that are adjacent to at least two rectangles (resp. vertices) among u_1, \dots, u_a , we call them *critical*.

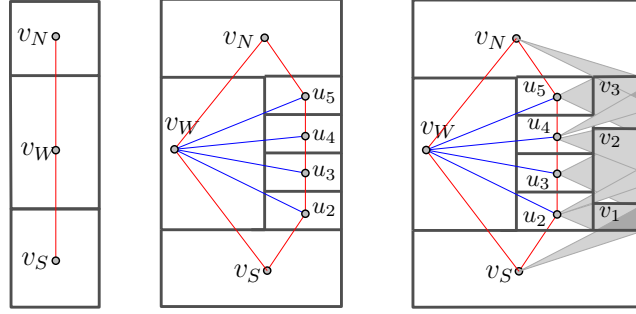


Fig. 10. Left: the base case. Middle and right: induction step illustrated on the graph of Figure 6. Before induction step we have simultaneous drawing of G_2 and D_2 (middle). After induction step we have simultaneous drawing of G_3 and D_3 (Figure 13). Intermediate steps are also shown in Figure 11. Middle: rectangles u_1, \dots, u_a appear on the right boundary of D_i . Right: placement the rectangles v_1, \dots, v_b . The light grey regions show the visibility of u_1, \dots, u_a inside v_1, \dots, v_b . The dark grey regions show the intersection of visibility regions of all neighbors.

We next show that the rectangles of the right boundary of D_{i+1} can be resized so that the gate of each critical rectangle is in the visibility region of each of its neighbor. The resizing consists of two modifications; we first stretch the right boundary of D_{i+1} , which ensures visibility to the gates of some of the critical vertices, and fulfillment of a special condition for the gates of the remaining critical vertices. We then obtain visibility to the gates of these remaining critical vertices by moving vertically some common boundaries of v_1, \dots, v_b , such that the existing visibilities are preserved. We do not perform any operation for non-critical vertices. Their placement is simple and will be explained at the end of the construction.

We first further specify the positions of the gates of those of v_1, \dots, v_b which are critical. See Figure 12, left for the illustration. Consider a critical vertex v_q , $1 \leq q \leq b$, and let p be the minimum index and ℓ be the maximum index such that $1 \leq p \leq \ell \leq a$ and u_p and u_ℓ are adjacent to v_q . Positions of the vertices u_p and u_ℓ are known by induction hypothesis. We place $\text{gate}(v_q)$ so that $y(u_p) < y_1(\text{gate}(v_q)) < y_2(\text{gate}(v_q)) < y(u_\ell)$. The reason for this positioning of the gates for the critical vertices will become clear later in the proof. The gates of the non-critical vertices are not of any interest to us, since they will not be used.

Consider a v_q , $1 \leq q \leq b$, and its neighbor, say u_p , $1 \leq p \leq a$. If u_p is a diverging neighbor of v_q then Statement 2 applies and determines the value $X(u_p, v_q) \geq x_2(v_q)$, such that $\forall x \geq X(u_p, v_q)$, if we set $x_2(v) = x$ then $\text{gate}(v_q) \subset \text{vis}(u_p, v_q)$. If u_p is a non-diverging neighbor of v_q , but $\text{gate}(v_q) \cap \text{blind}(u_p, v_q) = \emptyset$ then Statement 3 or Statement 4 determine the value $X(u_p, v_q) \geq x_2(v_q)$, such that $\forall x \geq X(u_p, v_q)$, if we set $x_2(v) = x$ then $\text{gate}(v_q) \cap \text{blind}(u_p, v_q) \neq \emptyset$. Finally, the values $X(u_1, v_1)$ and $X(u_a, v_b)$ are determined by Statement 1. Let $X = \max\{X(u_p, v_q) | 1 \leq p \leq a, 1 \leq q \leq b\}$.

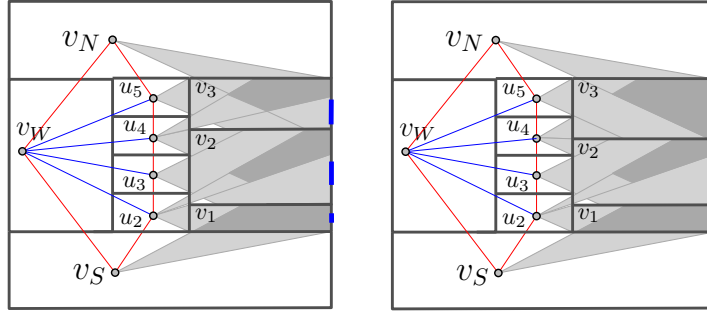


Fig. 11. Left: After the horizontal stretch of the right boundary of D_3 , where the initial D_3 is shown in Figure 10, right. The value of the stretch is determined by the pairs (v_N, v_3) , (u_4, v_2) and (u_2, v_2) . For the pair (v_N, v_3) (resp. (u_4, v_2)) Statement 1 (resp. Statement 2) is applied, to achieve that $\text{gate}(v_3) \subset \text{vis}(v_N, v_3)$ (resp. $\text{gate}(v_2) \subset \text{vis}(u_4, v_2)$). Finally, Statement 4 applies for the pair (u_2, v_2) , to archive that $\text{gate}(v_2) \cap \text{blind}(u_2, v_2) \neq \emptyset$. The remaining adjacencies does not increase the value of the stretch. Right: Statement 4 is applied to the pair (u_4, v_3) to archive that $\text{gate}(v_3) \subset \text{vis}(u_4, v_3)$ and as a result the common boundary $[v_2, v_3]$ is moved down.

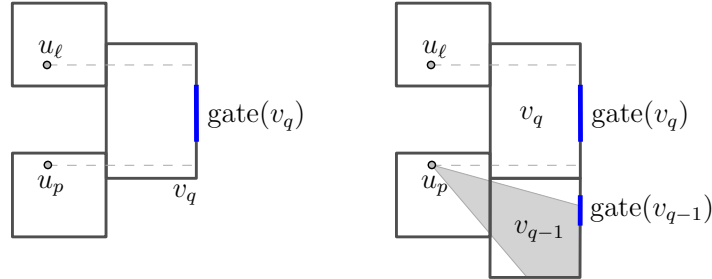


Fig. 12. Left: Illustration for the placement of gates of the critical vertices. Right: After second modification it holds that $\text{gate}(v_q) \subset R(v_q)$.

We now stretch the right boundary of D_{i+1} to have coordinate X (refer to Figure 11, left). By Statement 2, if u_p , $1 \leq p \leq a$, is a diverging neighbor of v_q , $1 \leq q \leq b$, then $\text{gate}(v_q) \subset \text{vis}(u_p, v_q)$.

Let now v_q , $1 \leq q \leq b$, be a vertex with non-diverging neighbor u_p , $1 \leq p \leq a$. The previous modification ensures that $\text{gate}(v_q) \cap \text{blind}(u_p, v_q) \neq \emptyset$. By Statements 3 and 4, there exists $Y(u_p, v_q) < y(u_p)$ (case $y(u_p) > y_2[u_p, v_q]$) or $Y(u_p, v_q) > y(u_p)$ (case $y(u_p) < y_1[u_p, v_q]$) such that if we set $y_2(v_q) = Y(u_p, v_q)$ then $\text{gate}(v_q) \subset \text{vis}(u_p, v_q)$ (refer to Figure 11, right and to Figure 13 for the specific example, refer also to Figure 14 for an abstract example).

In the following we show that the second modification does not destroy the visibilities to the critical vertices which existed after the first modification. First, observe that after the second modification the $\text{gate}(v_q)$, $1 \leq q \leq b$ of a

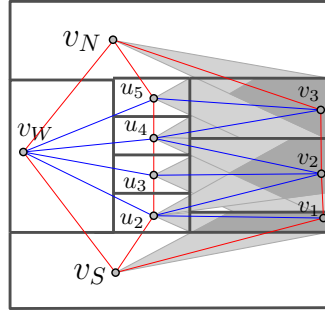


Fig. 13. Statement 4 is applied to the pair (u_2, v_2) to archive that $\text{gate}(v_2) \subset \text{vis}(u_2, v_2)$ and as a result the common boundary $[v_1, v_2]$ is moved down.

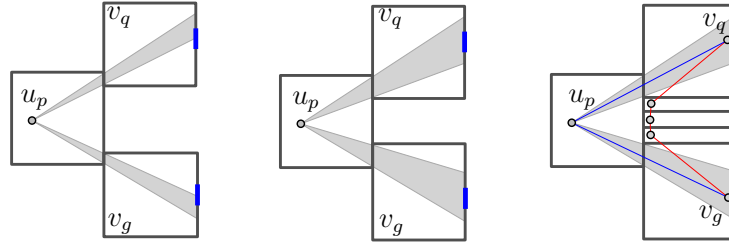


Fig. 14. Illustration of the second modification, where the common horizontal boundaries of v_1, \dots, v_b are possibly moved. Left: Vertex u_p is a non-diverging neighbour of v_q and v_g , it holds that $\text{blind}(u_p, v_q) \cap \text{gate}(v_q) \neq \emptyset$ and $\text{blind}(u_p, v_g) \cap \text{gate}(v_g) \neq \emptyset$. Middle: Statement 3 is applied to u_p and v_q . Statement 4 is applied to u_p and v_g . Right: the non-critical neighbors of u_p are placed and the vertices of G are positioned as explained in the proof.

critical vertex v_q still belongs to $R(v_q)$. This is ensured by the initial placement of the gates of the critical vertices. Thus, consider Figure 12 (right), where u_p is a non-diverging neighbor of v_{q-1} . The result of the second modification will be that the common boundary $[v_q, v_{q-1}]$ is moved up to $y < y(u_p)$. By the placement of the gate(v_q) we have that $y_1(\text{gate}(v_q)) > y(u_p)$ and therefore $y_1(\text{gate}(v_q)) > y$. By a symmetric argument for v_q, v_{q+1} and u_ℓ we infer that $\text{gate}(v_q) \subset R(v_q)$.

Second, assume that a vertex u_p is a non-diverging neighbor of v_q such that $y(u_p) < y_1[u_p, v_q]$ and such that $\text{gate}(v_q) \cap \text{blind}(u_p, v_q) \neq \emptyset$ (Figure 14), then the application of Statement 3 results in moving the segment $[v_q, v_{q-1}]$ down to a y -coordinate $Y(u_p, v_q) > y(u_p)$. The last inequality ensures that the visibility of u_p inside v_{q-1} has not changed.

We now explain how to draw the non-critical vertices, consider again Figure 14. Assume that u_p is a non-diverging neighbour of v_q , such that $y(u_p) < y_1[u_p, v_q]$ and v_g , such that $y(u_p) > y_1[u_p, v_g]$. As already mentioned, the application of Statement 3 results in moving the segment $[v_q, v_{q-1}]$ down to a y -

coordinate $Y(u_p, v_q) > y(u_p)$. The application of Statement 4 results in moving $[v_g, v_{g+1}]$ up to $Y(u_p, v_g) < y(u_p)$. Thus, $y[v_g, v_{g+1}] < y(u_p) < y[v_{p-1}, v_p]$ and the space between $y[v_g, v_{g+1}]$ and $y[v_{p-1}, v_p]$ is used for the placement of the non-critical neighbors of u_p (see Figure 14, right). It may happen that $v_{g+1} = v_p$, then we set $y_2(v_g) = y_1(v_q) = y(u_p)$.

Next, we place the actual vertices of the right boundary of f_{i+1} , except of u_1 and u_a , which have already been placed by induction hypothesis. For each critical vertex v_q we place vertex v_q very close to the middle of $\text{gate}(v_q)$. Since for each neighbor u_p of v_q $\text{gate}(v_q) \subset \text{vis}(u_p, v_q)$, the straight-line edge (u_p, v_q) crosses only $[u_p, v_q]$. A non-critical vertex v_q , which is adjacent to a single vertex u_p , is placed arbitrarily close to the common segment $[u_p, v_q]$. Thus, the straight-line edge (u_p, v_q) crosses only $[u_p, v_q]$. Edges (u_1, v_1) and (v_b, u_a) cross the segments $[u_1, v_1]$ and $[v_b, u_a]$, respectively, as ensured by the application of Statement 1. Finally, each edge (v_j, v_{j+1}) , $1 \leq j \leq b-1$, crosses only $[v_j, v_{j+1}]$, since $x_1(v_j) = x_1(v_{j+1})$ and $x_2(v_j) = x_2(v_{j+1})$. This concludes the proof of the theorem. \square

5 Conclusion

In this paper we considered the problem of drawing simultaneously a planar graph and its rectangular dual. We required that the vertices of the primal are positioned in the corresponding rectangles, the drawing of the primal graph is planar and straight-line, and each edge of the primal crosses only the rectangles where its end-points lie. Our proof is constructive and leads to a linear-time algorithm. However, the vertices are not placed on the grid and the area requirements of the construction are unclear. It would be interesting to either refine the algorithm to produce a simultaneous drawing with polynomial area, or to construct a counterexample, requiring an exponential area.

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